Bisector Graphs for Min-/Max-Volume Roofs
Over Simple Polygons

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Motivation

- Comparing two polygons. A lower area does not always lead to a lower roof volume.
- The lower envelope over all planes is not the minimum volume roof. (Neither does the upper envelope lead to the maximum volume roof.)
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**Approach**

- Building on *Roof Model and Bisector Graphs*[^2].
- *Wavefront Propagation*[^1] extended by two additional events.


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- Building on *Roof Model* and *Bisector Graphs*\(^2\).
- *Gradient Property*\(^2\) generalized.
- *Wavefront Propagation*\(^1\) extended by two additional events.

**Theorem (Roof ⇔ Bisector Graph\(^2\))**

*Every roof for P corresponds to a unique bisector graph of P, and vice versa.*

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- *Wavefront Propagation*[^1] extended by two additional events.

NATURAL GRADIENT PROPERTY

Let $\mathcal{R}(\mathcal{P})$ be a roof for $\mathcal{P}$. We say that a facet $f$ of $\mathcal{R}(\mathcal{P})$ has the *natural gradient property* if, for every point $p \in f$, there exists a path that (i) starts at $p$, (ii) follows the steepest gradient, and (iii) reaches the boundary of $\mathcal{P}$.


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**Extended Wavefront Propagation**

- Edge Event and Split Event\[^2\].
- Create Event and Divide Event

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**General Position**

- No two edges of $\mathcal{P}$ are parallel to each other.
- Not more than three bisectors of edges of $\mathcal{P}$ meet in one point.


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**Definition (Min-/Max-Volume Bisector Graph)**

The *maximum-volume bisector graph* $B_{\text{max}}(\mathcal{P})$ of a polygon $\mathcal{P}$ is a bisector graph $B(\mathcal{P})$ where the associated roof $R(\mathcal{P})$ has the natural gradient property for each of its facets and that maximizes the volume over all possible natural roofs for $\mathcal{P}$. Similarly for the *minimum-volume bisector graph* $B_{\text{min}}(\mathcal{P})$.


Two consecutive edges $e_i, e_j$ of $\mathcal{P}$. 
Edges of $\mathcal{P}$ are oriented. A half plane $\Pi(e)$ that starts at the supporting line $\ell(e)$ of an edge spans to its left. $\Pi(e)$ overlaps locally with the interior of $\mathcal{P}$.
A bisector $b_{i,j}$ spans from the intersection of the supporting line of two edges into their common interior.
Wavefront propagation of $e_i$ and $e_j$. 
Wavefront propagation of $e_i$ and $e_j$. A wavefront edge moves at unit speed (self parallel). The speed $s(v)$ of a wavefront vertex $v$ depends on the angle between the supporting lines forming its bisector\textsuperscript{[3]}.

$$s(v_{i,j}) = \frac{1}{\sin(\alpha/2)}$$
Every bisector defines a vertex that has a starting point and associated speed. In case such a vertex is not part of the wavefront we call it *stealth vertex*.

\[
s(v_{i,j}) = \frac{1}{\sin(\alpha/2)}
\]

\[
s(v_{j,k}) = \frac{1}{\sin(\beta/2)}
\]
Another input edge $e_x$ of $\mathcal{P}$.
At some point $p_{i,j,x}$ is the wavefront vertex incident with the supporting line from the wavefront edge of $e_x$. 
At some point $p_{i,j,x}$ is the wavefront vertex incident with the supporting line from the wavefront edge of $e_x$. The three bisectors meet at that point as well.
The wavefront changes: an additional edge $e$ is created, and $e$ is parallel to the wavefront edge of $e_x$. The two wavefront vertices on $b_{i,x}$ and $b_{j,x}$ are both reflex.
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Consecutive edges along a polygon boundary.
Consecutive edges along a polygon boundary. Wavefront propagation on the first (edge) event.
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The stealth vertex $v_{i,j}$ becomes incident with the wavefront edge originating from $e_x$ at point $p_{i,j,x}$. 
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The stealth vertex \( v_{i,j} \) becomes incident with the wavefront edge originating from \( e_x \) at point \( p_{i,j,x} \). Three arcs start at this point and create two new facets. One of these facets lies in the plane \( \Pi(e_i) \) and one in \( \Pi(e_j) \).
A small disc $c$ centered around a create event $p$ is partitioned into three wedges by the three arcs incident at $p$. If one wedge has an angle greater than $\pi$ it involves a wavefront vertex, starting at $p$, that moves faster than the wavefront vertex which ends at $p$. 
A small disc $c$ centered around a create event $p$ is partitioned into three wedges by the three arcs incident at $p$. If one wedge has an angle greater than $\pi$ it involves a wavefront vertex, starting at $p$, that moves faster than the wavefront vertex which ends at $p$. 
The wavefront propagation is used both to compute $B_{\min}(P)$ and $B_{\max}(P)$. The complexity is dominated by the computation of the create events. One create event takes $O(n \log n)$ time to compute and enqueue. There can be up to $O(n^2)$ create events.
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The overall complexity to compute $B_{\text{min}}(P)$ or $B_{\text{max}}(P)$ is in $O(n^3 \log n)$. 
Thanks for your attention!
The number of facets $B_{\text{min}}$ and $B_{\text{max}}$ can have is in $\mathcal{O}(n^2)$.
Lemma

The upper envelope of two natural roofs is not necessarily a natural roof.
